Solution (Number of Divisors)

Let $d(n)$ denoted the number of positive divisors of n. Note $d(n)$ is odd if and only if n is a perfect square. So let $a = \alpha^2, b = \beta^2$ and $c = \gamma^2$. Then the equation transforms to $\alpha^2+\beta^2 = \gamma^2$. Hence (α, β, γ) must be a Pythagorean triple. Now, note that $d(n) = 3$ only when n is a square of a prime number. Thus two of α, β, and γ must be prime numbers. Now, the first few Pythagorean triples are $(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), \ldots.$ Since a is larger than 50, thus $\alpha > 7$. Hence $(11, 60, 61)$ is the triple we are looking for. Therefore $(a, b, c) = (\alpha^2, \beta^2, \gamma^2) = (121, 3600, 3721)$.

Solution (Evaluate the sum)

First note that $a = 1$ is a solution to the first equation. Hence $S = 1 + 1 + 1^2 + \cdots + 1^{2011} = 2012$. Next, observe that

$$(a-1)S = (a-1)(1 + a + a^2 + \cdots + a^{2011}) = a^{2012} - 1$$

Then by rearranging the equation $a^{2012} - 7a + 6 = 0$, we have $a^{2012} - 1 = 7a - 7$. Thus

$$(a-1)S = a^{2012} - 1 = 7a - 7$$

Therefore $S = 7$ given that $a \neq 1$. So the only two possible values of S are 7 and 2012.